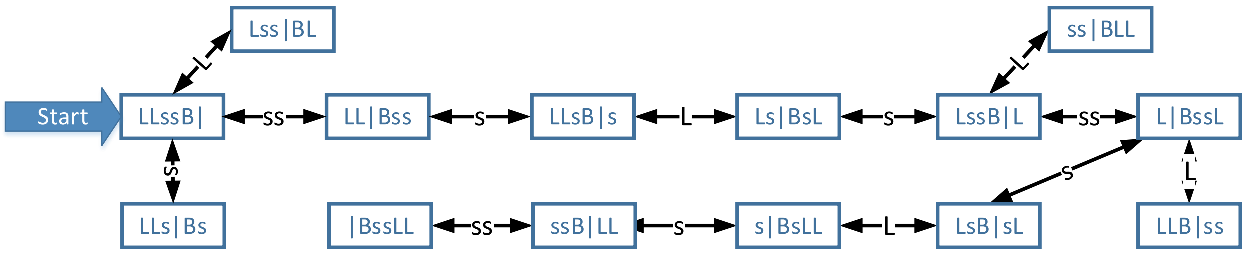
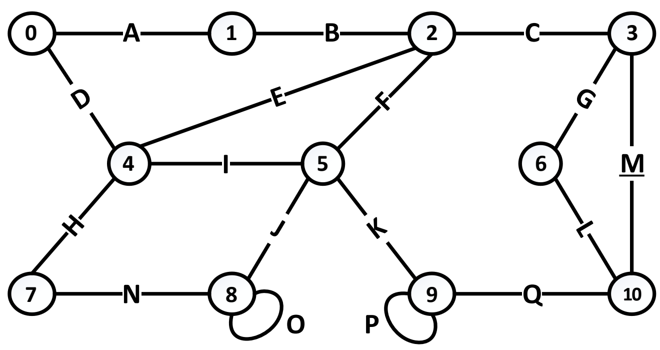
PART A – GRAPH THEORY – 25 MARKS

1. River Crossing (8 marks)



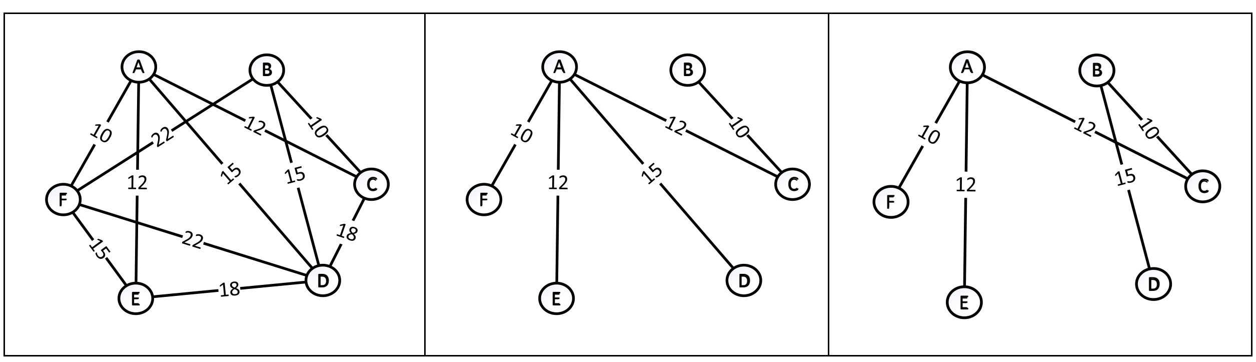
2. Euler and Hamiltonian Circuits (6 marks)



a) This graph does not have a **Euler** because at least one of the vertices has odd degree: both 3 and 10 have odd degree.
b) This graph has two **Hamiltonian** circuits starting at vertex 0:
- 0A1B2C3G6L10Q9K5J8N7H4D0
- 0D4H7N8JK9Q10L6G3C2B1A0

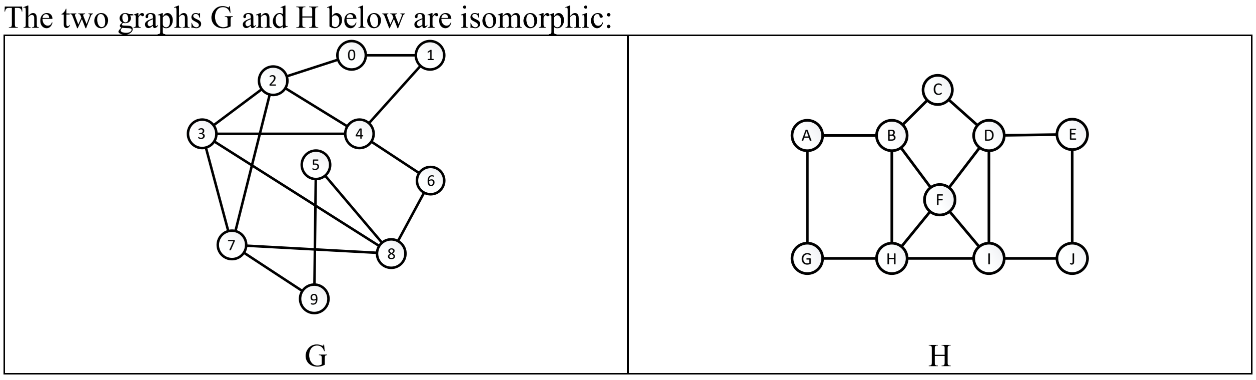
3. Minimum Spanning Tree (5 marks)

The graph in the left box below has two MSTs shown in the two right boxes



4. Isomorphic Graphs (6 marks)

The two graphs G and H below are isomorphic:



From V(G):	0	1	2	3	4	5	6	7	8	9
To V(H):	G	A	H	F	B	E	C	I	D	J
To V(H):	J	E	I	F	D	A	C	H	B	G

PART B – SEQUENCES, RECURRENCE RELATIONS – 10 MARKS

Given the sequence a_n defined with the recurrence relation:

$a_0 = 1$
$a_k = 3a_{k-1} + 7k$ for $k \geq 1$

1. Terms of the Sequence (5 marks)

$a_1 = 3.1 + 7.1$
$a_2 = 3(3.1 + 7.1) + 7.2 = 3^2 + 3.7.1 + 7.2$
$a_3 = 3(3^2 + 3.7.1 + 7.2) + 7.3 = 3^3 + 3^2.7.1 + 3^1.7.2 + 3^0.7.3$
$a_4 = 3(3^3 + 3^2.7.1 + 3^1.7.2 + 3^0.7.3) + 7.4 = 3^4 + 3^3.7.1 + 3^2.7.2 + 3^1.7.3 + 3^0.7.4$
$a_5 = 3(3^4 + 3^3.7.1 + 3^2.7.2 + 3^1.7.3 + 3^0.7.4) + 7.4 = 3^5 + 3^4.7.1 + 3^3.7.2 + 3^2.7.3 + 3^1.7.4 + 3^0.7.4$

2. Iteration (5 marks)

$a_n = 3^n + 7\sum_{i=0}^{n-1} 3^i(n-i)$

PART C – INDUCTION – 15 MARKS

Given the sequence b_n defined recursively as:

$b_0 = 1$, $b_1 = 4$
$b_k = 2b_{k-1} – b_{k-2}$ for $k \geq 2$

You will now prove by **strong induction** that a solution to this sequence is $b_n = 1+3n$.

1. Problem Statement (2 marks)

The conjecture being proved is expressed symbolically in the form $\forall n \in D$, P(n), where:
a) (1 mark) $D = \mathbb{N}$
b) (1 mark) P(n) is: $b_n = 1+3n$

2. Base Cases (4 marks)

- When n=0, $1+3n = 1+3.0 = 1 = b_0$
- When n=1, $1+3n = 1+3.1 = 4 = b_1$

3. Inductive step setup (3.5 marks)

c) (2 marks) State the assumption in the inductive step and identify the inductive hypothesis.
Assume that some $k \geq 1$ is such that $\forall m \in \{0,\ldots,k\}$ $b_m = 1+3m$ ← IH: Inductive Hypothesis
d) (1.5 marks) State what you will be proving in the inductive step.
We will prove P(k+1), i.e. $b_{k+1} = 1+3(k+1) = 3k+4$

4. Remainder of Inductive step (5.5 marks).

Since $k \geq 1$ then $k+1 \geq 2$ and the recurrence relation applies to k+1:
$b_{k+1} = 2b_k – b_{k-1}$
$k \leq k$ and $k-1 \leq k$ and therefore the inductive hypothesis applies to them:
$b_{k+1} = 2(1+3k) – (1+3(k-1)) = 2+6k-1-3k+3 = 3k+4$ By algebra
QED